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Engineering Notes

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Extremal Angle of Attack over a Singular Thrust Arc in Rocket Flight

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Introduction

IN determining optimal trajectories for atmospheric rocket flight, as in the case of synergetic maneuvers, it is quite common to approximate the thrust vector to be aligned with the velocity vector.[1–3] The ensuing dynamical model results in some simplification of the optimal control problem. This approximation is justified with the assumption that the angle of attack is small and that the engine is rigidly fixed to the vehicle. For higher angles of attack (and a body-fixed thrust vector), one cannot make this assumption and, consequently, the problem of determining the optimal angle of attack is no longer independent of the problem of determining the optimal thrust vector.[1] Consequently, the optimal thrust steering is not necessarily given by the primer vector as is the case in exoatmospheric flight.[1,4] Studies of the atmospheric portion of the synergetic maneuver called aerocruise indicate that the optimal angle of attack may be quite high,[5] even higher than the angle of attack at maximum lift-to-drag ratio. However, the thrust program employed by the aerocruise maneuver is not optimal,[3,6] and thus the coupled problem of determining the optimal angle of attack and the optimal thrust program without the limitation of the small-angle assumption is still an open problem. For the purpose of analysis, this problem may be classified as either regular or singular.[7–9] The problem becomes singular due to the possibility of thrust modulation, and it occurs often enough[1–4,8] that it warrants a careful study. In this Note, we analyze the singular problem and show that the extremal angle of attack is given by a very simple transcendental equation.

Problem Formulation

The main problem is to determine the optimal control program $\boldsymbol{u}^*(\cdot)$ that maneuvers an aerodynamic rocket vehicle from a given initial condition, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, to a target manifold, $\boldsymbol{y}(\boldsymbol{x}_f, t_f) = \boldsymbol{0}$, $\boldsymbol{y} \in \mathcal{R}^s$, $s \leq 7$, while minimizing a Mayer performance index,

$$J[\boldsymbol{u}] = F(\boldsymbol{x}_f, t_f) \tag{1}$$

where $\boldsymbol{x}$ is the seven-dimensional state vector $\boldsymbol{x} = [r, \theta, \varphi, V, \psi, \gamma, m]^T$ that consists of the radial position, longitude, latitude, speed, heading angle, flight-path angle, and mass of the spacecraft, respectively. The subscripts 0 and f denote initial and final conditions, respectively. The control vector $\boldsymbol{u} = [\alpha, \delta, T]^T$ consists of the angle of attack, bank angle, and thrust magnitude, respectively, with some specified bounds that are stipulated as $\boldsymbol{u} \in U$, where U is the set of allowable controls. In the analysis that follows, an explicit formulation of the boundary conditions and performance index are not necessary because the results remain valid for the entire class of problems.

The dynamics of the vehicle (in a nonrotating atmosphere) are given by[1]

$$\dot{r} = V \sin\gamma \tag{2a}$$

$$\dot{\theta} = \frac{V \cos\gamma \cos\psi}{r \cos\varphi} \tag{2b}$$

$$\dot{\varphi} = \frac{V \cos\gamma \sin\psi}{r} \tag{2c}$$

$$\dot{V} = a_s - g \sin\gamma \tag{2d}$$

$$\dot{\psi} = \frac{a_w}{V \cos\gamma} - \frac{V}{r} \cos\gamma \cos\psi \tan\varphi \tag{2e}$$

$$\dot{\gamma} = (a_n/V) - [g - (V^2/r)](\cos\gamma/V) \tag{2f}$$

$$\dot{m} = -(T/v_e) \tag{2g}$$

where $g = \mu/r^2$ is the inverse-square gravitational acceleration, v_e is the rocket exhaust speed, and a_s, a_n, and a_w are the (nongravitational) perturbing accelerations on the vehicle in the tangential, normal, and binormal directions, respectively, given by

$$a_s = \frac{T \cos\alpha - D}{m} \tag{3a}$$

$$a_n = \frac{(L + T \sin\alpha) \cos\delta}{m} \tag{3b}$$

$$a_w = \frac{(L + T \sin\alpha) \sin\delta}{m} \tag{3c}$$

where L and D are the aerodynamic lift and drag, respectively. In the analysis to follow, it suffices to assume that $L = L(\boldsymbol{x}, \alpha)$ and $D = D(\boldsymbol{x}, \alpha)$. Note that the thrust vector is assumed to be fixed along the reference body axis of the vehicle from which α is measured. It is important to emphasize that an independent thrust vector control (i.e., a gimbaled system) is not considered here.

First-Order Necessary Conditions

The so-called first-order necessary conditions[9,10] are those that are derivable by the Maximum Principle. The Hamiltonian is given by

$$\begin{aligned} H = {} & \lambda_r V \sin\gamma + \lambda_\theta \frac{V \cos\gamma \cos\psi}{r \cos\varphi} + \lambda_\varphi \frac{V \cos\gamma \sin\psi}{r} \\ & - \lambda_V g \sin\gamma - \lambda_\psi \frac{V}{r} \cos\gamma \cos\psi \tan\varphi - \lambda_\gamma \left(g - \frac{V^2}{r}\right) \frac{\cos\gamma}{V} \\ & + \lambda_V \frac{T \cos\alpha - D}{m} + \lambda_\psi \frac{(L + T \sin\alpha) \sin\delta}{mV \cos\gamma} \\ & + \lambda_\gamma \frac{(L + T \sin\alpha) \cos\delta}{mV} - \lambda_m \frac{T}{v_e} \end{aligned} \tag{4}$$

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where the λ_r, λ_θ, λ_φ, λ_V, λ_ψ, λ_γ, and λ_m are the components of the costate $\boldsymbol{\lambda}$, which satisfies

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{x}} \tag{5}$$

For the purpose of brevity, the partial derivatives are not carried out explicitly. Applying the Maximum Principle for the special case of interior controls (i.e., when $\boldsymbol{u}^* \in \operatorname{int} U$, $\partial H/\partial \boldsymbol{u} = \boldsymbol{0}$) yields

$$\frac{\partial H}{\partial \alpha} = -\lambda_V \frac{T\sin\alpha + D_\alpha}{m} + \left(\frac{L_\alpha + T\cos\alpha}{mV}\right)\left(\lambda_\psi \frac{\sin\delta}{\cos\gamma} + \lambda_\gamma \cos\delta\right) = 0 \tag{6}$$

$$\frac{\partial H}{\partial \delta} = \frac{L + T\sin\alpha}{mV}\left(\lambda_\psi \frac{\cos\delta}{\cos\gamma} - \lambda_\gamma \sin\delta\right) = 0 \tag{7}$$

$$\frac{\partial H}{\partial T} = \lambda_V \frac{\cos\alpha}{m} + \lambda_\psi \frac{\sin\alpha\sin\delta}{mV\cos\gamma} + \lambda_\gamma \frac{\sin\alpha\cos\delta}{mV} - \lambda_m \frac{1}{v_e} = 0 \tag{8}$$

where L_α and D_α are the partial derivatives of lift and drag with respect to the angle of attack. Note that these conditions and those to follow hold for the optimal controls and, hence, the drop in the asterisk notation. The last equation demonstrates that an optimal thrust modulation means that it is singular control (i.e., $\partial^2 H/\partial T^2 = 0$ over a nonzero time interval). Barring this possibility, the optimal thrusting follows the bang-bang principle and one only needs to determine the switching structure. However, as mentioned before, the singular problem should be studied as part of the totality of extremal arcs. For the remainder of this Note, we will assume that $\partial^2 H/\partial\alpha^2 \neq 0$ and $\partial^2 H/\partial\delta^2 \neq 0$, in which case the resulting extremal is a partially singular arc.[10] The strengthened Legendre–Clebsch condition on the bank angle ($\partial^2 H/\partial\delta^2 < 0$) gives us an important condition on λ_V that will be used later. Carrying out this second derivative, we arrive at

$$\lambda_\psi(\sin\delta/\cos\gamma) + \lambda_\gamma\cos\delta > 0 \tag{9}$$

which after combining with Eq. (6) yields (assuming that $L_\alpha + T\cos\alpha > 0$ and $T\sin\alpha + D_\alpha > 0$, a reasonable assumption for a large number of cases),

$$\lambda_V > 0 \tag{10}$$

Higher-Order Necessary Condition

Higher-order necessary conditions[9,10] are those that cannot be derived from the maximization of the Hamiltonian. Thus, although Eq. (8) may be differentiated repeatedly with respect to time to obtain an equation for the extremal thrust modulation, additional conditions are necessary to determine the optimality of the singular arc. As the resulting extremal is a partially singular arc, a powerful quadratic necessary condition derived by Goh[11] and Robbins,[12] which we call the Goh–Robbins condition (GRC), may be used to glean further information on the nature of the singular arc. The GRC may be expressed quite conveniently in terms of the Hamiltonian as[9]

$$\frac{\partial}{\partial u_i}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{\partial H}{\partial u_j} = 0 \tag{11}$$

for $k = 0, \ldots, (q_i + q_j - 1)$, where q_i and q_j are the orders of the controls u_i and u_j. For our problem, $\boldsymbol{u} = [\alpha, \delta, T]^T$ with $q_1 = q_2 = 0$ and $q_3 = 1$. Thus, $k = 0$, and the GRC reduces to

$$\frac{\partial}{\partial\alpha}\frac{\partial H}{\partial T} = 0 \tag{12}$$

$$\frac{\partial}{\partial\delta}\frac{\partial H}{\partial T} = 0 \tag{13}$$

Carrying out the indicated operations, we arrive at

$$\frac{\partial}{\partial\alpha}\frac{\partial H}{\partial T} = -\lambda_V\frac{\sin\alpha}{m} + \lambda_\psi\frac{\cos\alpha\sin\delta}{mV\cos\gamma} + \lambda_\gamma\frac{\cos\alpha\cos\delta}{mV} = 0 \tag{14}$$

$$\frac{\partial}{\partial\delta}\frac{\partial H}{\partial T} = \frac{\sin\alpha}{mV}\left(\lambda_\psi\frac{\cos\delta}{\cos\gamma} - \lambda_\gamma\sin\delta\right) = 0 \tag{15}$$

Equation (15) is automatically satisfied by the first-order necessary condition, Eq. (7), and yields no new information; however, Eq. (14) yields an additional necessary condition, which provides a critical piece of information. Rewriting Eq. (6), the first-order necessary condition for the optimal angle of attack as

$$\left(-\lambda_V\frac{\sin\alpha}{m} + \lambda_\psi\frac{\cos\alpha\sin\delta}{mV\cos\gamma} + \lambda_\gamma\frac{\cos\alpha\cos\delta}{mV}\right)T + \left(-\lambda_V\frac{D_\alpha}{m} + \lambda_\psi\frac{L_\alpha\sin\delta}{mV\cos\gamma} + \lambda_\gamma\frac{L_\alpha\cos\delta}{mV}\right) = 0 \tag{16}$$

it is clear that the first term in parenthesis must vanish due to the quadratic necessary condition, Eq. (14). Hence, we must have

$$-\lambda_V\frac{D_\alpha}{m} + \lambda_\psi\frac{L_\alpha\sin\delta}{mV\cos\gamma} + \lambda_\gamma\frac{L_\alpha\cos\delta}{mV} = 0 \tag{17}$$

Using Eq. (14) to eliminate λ_ψ and λ_γ, we arrive at

$$\lambda_V\left[-\frac{D_\alpha}{m} + \frac{L_\alpha\sin\alpha}{m\cos\alpha}\right] = 0 \tag{18}$$

The costate $\lambda_V \neq 0$; in fact, $\lambda_V > 0$ form the strengthened Legendre–Clebsch condition, Eq. (10). Thus Eq. (18) reduces to the very elegant condition

$$\tan\alpha = \frac{D_\alpha}{L_\alpha} = \frac{\partial C_D}{\partial\alpha}\bigg/\frac{\partial C_L}{\partial\alpha} \tag{19}$$

where C_L and C_D are lift and drag coefficients, respectively. For a given drag polar, this equation may be solved quite readily for the extremal angle of attack over all flight regimes. If one ignores the dependence of the aerodynamic coefficients over such quantities as Mach number and Reynolds number, then Eq. (19) yields a constant α program, which could serve as a useful guidance parameter.

Conclusions

The Goh–Robbins necessary condition for the optimality of angle of attack over a singular thrust arc leads to a simple angle-of-attack program given by an elementary transcendental relationship. This result was derived without an a priori assumption of a small angle of attack and thus represents a useful generalization that applies to any (interior) angle of attack. As this condition was derived with minimal assumptions (e.g., no assumption was made on the nature of the drag polar), it could be applied in many practical situations. It is, of course, necessary to determine the associated controls, namely, the singular thrust profile and the nonsingular bank angle, which is a subject of ongoing research. Thereafter, the results may be applied to a specific problem with a given performance index and boundary condition.

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Gravity Turn Descent with Quadratic Air Drag

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Introduction

GRAVITY turn guidance has been widely investigated[1–3] and used in practice[4,5] for terminal descent to a planetary surface. The descent method requires that the vehicle thrust vector is aligned opposite to the vehicle velocity vector at all points along the descent trajectory. This may be easily implemented onboard by using the vehicle attitude control system to null body rates about the vehicle velocity vector. The method is also efficient, providing near minimum-fuel descents. Previous studies of the method have assumed that the descent maneuver takes place in a vacuum. However, some scenarios, such as descent to a planetary surface or moon with a significant atmosphere, may lead to more complex descent dynamics.

In this Note the gravity turn maneuver is considered with quadratic air drag. It is assumed that the terminal maneuver is initiated at low altitude, where the air density and vehicle drag coefficient may be considered near constant. Because the gravity turn maneuver requires the vehicle thrust vector to be aligned opposite to the vehicle velocity vector, the angle of attack will also be constant. It is shown that the modified equations of motion have a closed-form solution obtained from Bernoulli's equation.

Vacuum Gravity Turn

First, the standard gravity turn maneuver will be reviewed and the equations of motion solved in closed analytical form. For terminal descent maneuvers, the vehicle planar translational dynamics may be modeled as a point mass moving over a flat surface at altitude h with a uniform local gravitational acceleration g, viz.,

$$\dot{v} = -\alpha g + g\cos\psi \tag{1a}$$

$$v\dot{\psi} = -g\sin\psi \tag{1b}$$

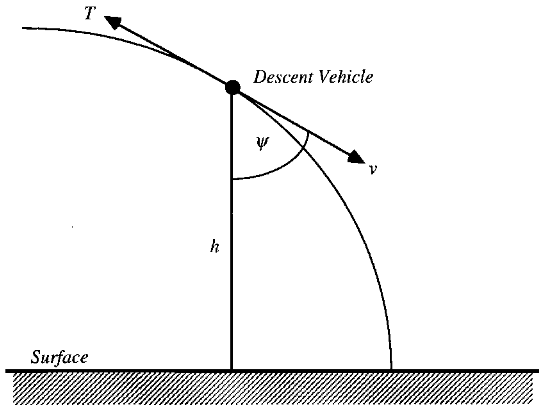

Fig. 1 Schematic gravity turn descent.

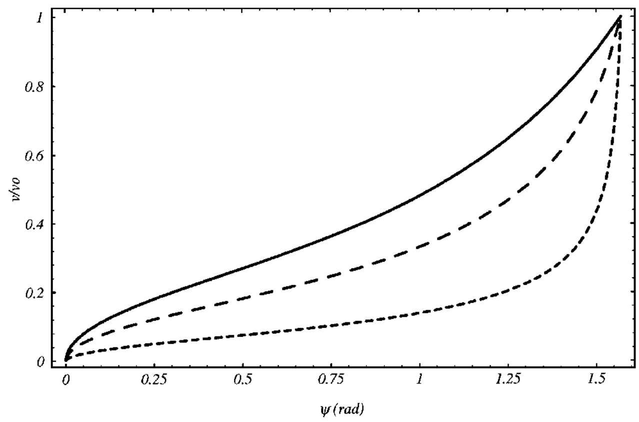

Fig. 2 Velocity-flight-path-angle profile:——, vacuum; — — —, $\beta v_0^2 = 10^{-3}$; and - - - -, $\beta v_0^2 = 10^{-2}$.

where the state variables are illustrated in Fig. 1. The thrust-weight ratio $\alpha = T/mg$ is assumed to be constant during the descent maneuver, although throttling may be used to track defined descent contours.[6]

The preceding equations may now be solved by using the flight-path angle ψ as the independent variable. Then, Eqs. (1) yield a single first-order equation, viz.,

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\psi} = \alpha\,\mathrm{cosec}\,\psi - \cot\psi \tag{2}$$

The solution for the vehicle velocity as a function of flight-path angle follows directly by integration and may be written as[1]

$$v(\psi) = v_0\left[\frac{\cos(\psi/2)}{\cos(\psi_0/2)}\right]^{-(1+\alpha)}\left[\frac{\sin(\psi/2)}{\sin(\psi_0/2)}\right]^{-(1-\alpha)} \tag{3}$$

A gravity turn descent with $\alpha = 1.5$ is shown in Fig. 2. It can be seen that the maneuver has the useful property that a vertical landing is ensured since $v \to 0$ as $\psi \to 0$. To ensure that $v \to 0$ as $h \to 0$ requires an appropriate choice of α for a given initial altitude.[1]

Gravity Turn with Quadratic Air Drag

The gravity turn maneuver is now reconsidered with the addition of quadratic air drag. It will be demonstrated that the equations of motion can again be solved in closed form and that the solutions reduce to the vacuum case derived earlier. Because the thrust vector is oriented opposite the velocity vector, air drag augments the

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